Linear Maps on Symmetric Spaces: The Invariance of Nonzero Decomposable Elements

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ABSTRACT

Let U be an n-dimensional vector space over an algebraically closed field of characteristic zero. We show that every linear mapping on the rth symmetric product space over U that preserves nonzero decomposable elements is induced by a nonsingular linear mapping on U when $2 < n \leqslant r$.

1. INTRODUCTION

Let U be a finite dimensional vector space over a field F. Let $U^{(r)}$ denote the rth symmetric product space over U with decomposable elements denoted by $x_1 \cdots x_r$ where $x_1, \ldots, x_r \in U$. A linear mapping on $U^{(r)}$ is called a *decomposable mapping* if it sends nonzero decomposable elements to nonzero decomposable elements. If F is algebraically closed of characteristic zero and $\dim U \geqslant r+1$, it is known [4, 7] that every decomposable mapping T on $U^{(r)}$ is induced by a nonsingular linear mapping f on U; that is, $T(x_1 \cdots x_r) = f(x_1) \cdots f(x_r)$ for all $x_1, \ldots, x_r$ in U. In this note we prove that the same is true if $3 \leqslant \dim U \leqslant r$ by using some results concerning algebraic varieties.

Linear mappings on Grassmann product spaces which preserve decomposable elements were studied by Westwick in [12, 14, 16]. For a survey of this area of work see [9].

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2. RESULTS

Let F be a field. An algebraic variety in the affine n-space F^n is called *homogeneous* if it is defined by a set of homogeneous polynomials over F.

We begin by proving the following useful general result.

LEMMA 1. *Let F be algebraically closed. Let D be a homogeneous algebraic variety in F^n such that* $\dim D > \dim V$ *for any linear subspace V of F^n contained in D. If $T : F^n \to F^n$ is a linear mapping such that*

$$T(D-\{0\}) \subseteq D-\{0\},$$

then it is impossible that $\operatorname{Im} T \subseteq D$.

REMARK. In particular, the lemma is true if D is an irreducible homogeneous variety which is not a linear subspace of F^n.

Proof. Suppose the contrary that $\operatorname{Im} T \subseteq D$. Let E be an irreducible component of D of maximal dimension. Then E is homogeneous, since D is homogeneous [5, p. 35]. Hence $\operatorname{Ker} T \cap E = \{0\}$, since $T(D-\{0\}) \subseteq D-\{0\}$. By Proposition 3.8 in [5, p. 133], we have

$$\dim(\operatorname{Ker} T \cap E) \geqslant \dim \operatorname{Ker} T + \dim E - n.$$

This implies that

$$0 \geqslant (n - \operatorname{rank} T) + \dim D - n,$$

a contradiction, since $\dim D > \operatorname{rank} T$ by hypothesis. Hence D does not contain $\operatorname{Im} T$. ∎

Let U be a finite dimensional vector space over F. We have

THEOREM 1. *Let T be a decomposable mapping on $U^{(r)}$. If $2 < \dim U \leqslant r$ and F is algebraically closed of characteristic zero, then T is induced by a nonsingular linear mapping on U.*

Proof. By Theorem 2 in [7], we have either that T is induced by a nonsingular linear mapping on U or that $T(U^{(r)}) = W^{(r)}$ for some two dimensional subspace W of U.

Suppose the latter holds. Let $D=\{x_1\cdots x_r:\dim\langle x_1,\dots,x_r\rangle\leqslant 2\}$. Then it was shown in [11, p. 71] that D is a homogeneous algebraic variety in $U^{(r)}$. Let V be a two-dimensional subspace of U. Since F is algebraically closed, it follows that $V^{(r)}$ consists of decomposable elements (see [3]) and hence $V^{(r)}\subseteq D$. Thus $T(D-\{0\})\subseteq D-\{0\}$. Suppose that $\dim D=r+1$. Since $V^{(r)}$ is contained in some irreducible component E of D and $\dim V^{(r)}=r+1$, it follows that $V^{(r)}=E$. This shows that D has infinitely many irreducible components, a contradiction. Hence $\dim D>r+1$. Since $\dim U\leqslant r$, every subspace of $U^{(r)}$ contained in D has dimension $\leqslant r+1$ (see [3]). By Lemma 1 we obtain a contradiction. This proves that T is induced by a nonsingular linear mapping on U. ■

We now show that Lemma 1 could also be applied to study linear mappings on other vector spaces which preserve certain algebraic varieties.

In [14], Westwick proved the following result:

THEOREM 2. *Let T be a singular linear mapping on the rth Grassmann product $\bigwedge^r U$ which sends nonzero decomposable elements to nonzero decomposable elements. Then* $\operatorname{Im} T$ *consists of decomposable elements.*

Suppose that F is algebraically closed. Then the set of all decomposable elements in $\bigwedge^r U$ forms an irreducible homogeneous variety [6]. Hence Theorem 2 and Lemma 1 imply that every linear mapping S on $\bigwedge^r U$ which preserves nonzero decomposable elements is nonsingular. This was first proved in [12] by a different method and then used to determine the structure of S.

The following result was obtained by Westwick for $k=1$ [13] and by Chan and Lim for $k>1$ [2].

THEOREM 3. *Let k be a fixed positive integer. Let D_k be the set of all nonzero matrices of rank $\leqslant k$ in the space $M_{m,n}(F)$ of all $m\times n$ matrices over F, where $k<\min\{m,n\}$. Let F be infinite, and T a linear mapping on $M_{m,n}(F)$ such that $T(D_k)\subseteq D_k$. Then one of the following holds:*

(i) *There exist an $m\times m$ nonsingular matrix P and an $n\times n$ nonsingular matrix Q such that either $T(A)=PAQ$ for all A, or*

$$m=n \quad \textit{and} \quad T(A)=PA^tQ \quad \textit{for all} \quad A.$$

(ii) $\operatorname{Im} T$ *consists of matrices of rank* $\leqslant k$.

Suppose F is assumed to be algebraically closed in Theorem 3. Then $D_k \cup \{0\}$ is an irreducible homogeneous variety in $M_{m,n}(F)$ (see [15]). Hence Theorem 3 and Lemma 1 imply that (ii) cannot occur and hence T is of the form (i). This was proved by Marcus and Moyls [10] for $k = 1$ and by Beasley [1] for $k > 1$ using different methods.

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